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Expanded Third-Order Markov Undulation Model

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An expanded form of the third-order Markov undulation model is developed which includes undulation of the geoid, vertical deflections, gravity anomaly, and the anomalous gravity gradients. First, the complete set of autocorrelation and cross-correlation functions, as specified by the autocorrelation of undulation, is obtained. Then, a time-domain approach is used to synthesize the corresponding linear system driven by white noise that represents the anomalous gravity field behavior along a straight horizontal track. The resulting nine-state Markov model provides a shaping filter for applying covariance analysis and optimal estimation techniques to inertial navigation and gravity gradiometry systems analysis.

Introduction

CONTINUED mechanical refinement has brought inertial navigation technology to the stage where unmodeled variations in the Earth's gravity field have become significant and sometimes dominant error sources.[1] At the same time, inertial navigation system analysts have been facing the task of developing error models to evaluate the impact of gravity uncertainties on navigation system performance. Additionally, the development of gravity gradiometers has made it necessary to model gravity-gradient signals to specify sensor requirements and analyze gradiometer-aided inertial navigation or gradiometry survey performance.

To apply the powerful tools of linear system theory and Kalman filtering, it has become common to consider gravity uncertainties random variables, and to model them as Markov processes. Following the first treatment of vertical deflections as a first-order Markov process,[2] a succession of higher order Markov models was proposed culminating with the third-order Markov undulation model[3,4] which is fully self-consistent; that is, the auto- and cross-correlation functions of undulation, vertical deflections, and gravity anomaly obey and are derived according to the mathematical relationships of physical geodesy.

The motivation for using Markov models for gravity uncertainties is that they can be cast in the form of linear differential equations driven by white noise. This allows use of the model as a shaping filter in covariance analysis of linear dynamic systems, Kalman filters, and optimal smoothing algorithms. The expanded third-order Markov undulation model described in this paper is derived from the original version, but it has three differences.

First, it was found that the original model could easily be extended to include all the anomalous gravity gradients by continued differentiation of auto- and cross-correlation functions.

Second, the exact cross-correlation function between undulation and gravity anomaly contains modified Bessel functions.[4] Modified Bessel functions are solutions to a class of differential equations whose coefficients are functions of the independent variable. To obtain a Markov shaping filter, it is necessary to substitute an approximate function which is the solution to a linear differential equation with constant coefficients. While the chosen exponential form did not approximate the exact Bessel function as well as at first hoped possible, it provides a suitable representation of the undulation-gravity anomaly cross-correlation.

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Third, the synthesis of a set of white-noise-driven differential equations from the auto- and cross-correlation functions has been a difficult task for high-order models. Usually, power spectral density functions are obtained from the autocorrelation functions, and then factored into transfer functions for which a linear system can be specified.[5] In high-order models, sheer complexity causes this approach to become unmanageable. Cross-correlations must often be ignored, or are unintentionally lost. At times, arbitrary adjustments must be made. In this paper, a different approach to the synthesis problem is used, which for any order of system uniquely specifies the shaping filter. Only matrix manipulations are necessary, and the procedure has the potential to be completely computerized once the autocorrelation of undulation is given.

Auto- and Cross-Correlation Functions for the Expanded Third-Order Markov Undulation Model

The model is derived by proposing an autocorrelation function for geoidal undulation, which is equivalent to a description of the anomalous potential, and then deriving all other correlation functions from it, using stochastic relationships constrained by geodetic theory. The proposed autocorrelation function for undulation is[4]

$$\phi_{NN}(r) = \sigma_N^2 \left(1 + \frac{r}{d} + \frac{r^2}{3d^2}\right) e^{-r/d} \tag{1}$$

$$= \mathrm{E}[N(X,Y)N(X+x, Y+y)] \tag{2}$$

where r is radial horizontal distance $(=\sqrt{x^2+y^2})$, d is a distance parameter, N is geoidal undulation, σ_N is the rms value of undulation, and E[ ] indicates the expectation operation. From this function, it is possible to derive the complete set of correlation functions for vertical deflections, gravity anomaly, and the anomalous gravity gradients.

Coordinate Frame and Geodetic Relationships

A local vertical coordinate frame, shown in Fig. 1, is used. Then

$$\eta = -\partial N/\partial x \qquad \xi = -\partial N/\partial y \tag{3}$$

$$\Gamma_{xx} = -g\partial\eta/\partial x \qquad \Gamma_{yy} = -g\partial\xi/\partial y \tag{4}$$

$$\Gamma_{xy} = -g\partial\eta/\partial y = -g\partial\xi/\partial x \tag{5}$$

where N is geoidal undulation, η is the prime component of vertical deflection, ξ is the meridian component of vertical deflection, and Γ_{xx}, Γ_{xy}, Γ_{yy} are anomalous horizontal gravity

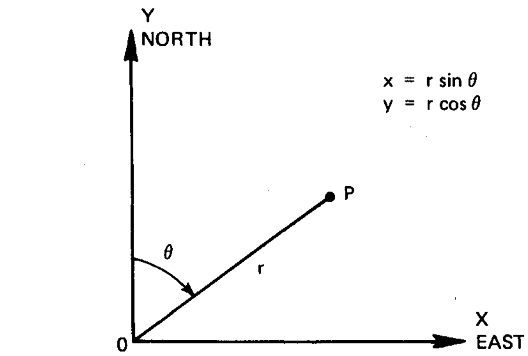

Fig. 1 Coordinate convention.

gradients. Also

$$\Gamma_{xz} = \partial \Delta g / \partial x \qquad \Gamma_{yz} = \partial \Delta g / \partial y \tag{6}$$

where Δg is the gravity anomaly, and Γ_{xz}, Γ_{yz} are anomalous vertical shear gravity gradients.

Derivation of Correlation Functions

Random variable theory[5] provides the relations for derivation of all other auto- and cross-correlation functions from Eq. (1). All but one can be obtained by taking successive partial derivatives. To obtain $\phi_{\Delta g \Delta g}(r)$ from $\phi_{NN}(r)$, the flat-Earth form of the Vening-Meinesz equations must be used to show[4]

$$\begin{aligned} \phi_{\Delta g \Delta g}(r) &= -g\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi_{NN}(r) \\ &= \sigma^2_{\Delta g}\left(1 + \frac{r}{d} - \frac{r^2}{2d^2}\right)e^{-r/d} \end{aligned} \tag{7}$$

Finding $\phi_{N\Delta g}(r)$ is mathematically involved, and requires two Fourier transform operations giving a solution of modified Bessel functions.[4] Since modified Bessel functions are not solutions of linear differntial equations with constant coefficients, and it is necessary to have all correlation functions of such a form to synthesize a nonvarying linear system, this function is approximated by

$$\phi_{N\Delta g}(r) \cong \frac{2\sigma_N \sigma_{\Delta g}}{\sqrt{6}}\left(1 + \frac{3r}{2d} + \frac{3r^2}{4d^2}\right)e^{-3r/2d} \tag{8}$$

as shown in Fig. 2. Attempts to fit the exact function with a third-order exponential that dipped below the r/d axis failed to produce a positive-definite covariance matrix. Whether a better approximation of some other suitable form exists is not known.

Now, it is possible to obtain all remaining correlation functions by taking successive partial derivatives of Eqs. (1), (7), and (8).[5] This procedure is facilitated by working in r, θ coordinates, and using the differential formulas

$$\frac{\partial \phi(r,\theta)}{\partial x} = \sin\theta \frac{\partial \phi(r,\theta)}{\partial r} + \frac{\cos\theta}{r}\frac{\partial \phi(r,\theta)}{\partial \theta} \tag{9}$$

$$\frac{\partial \phi(r,\theta)}{\partial y} = \cos\theta \frac{\partial \phi(r,\theta)}{\partial r} - \frac{\sin\theta}{r}\frac{\partial \phi(r,\theta)}{\partial \theta} \tag{10}$$

The derivation process can be organized into three "trees" of partial derivative operations, as shown in Figs. 3-5.

These trees are symmetric about their centerlines in the sense that one side can be obtained from the other simply by replacing $\sin\theta$ by $\cos\theta$, or $\cos\theta$ by $\sin\theta$, and changing the subscripts on the variances. For example, $\phi_{N\xi}(r,\theta)$ can be found from $\phi_{N\eta}(r,\theta)$ by replacing $\sin\theta$ with $\cos\theta$, and σ_η with

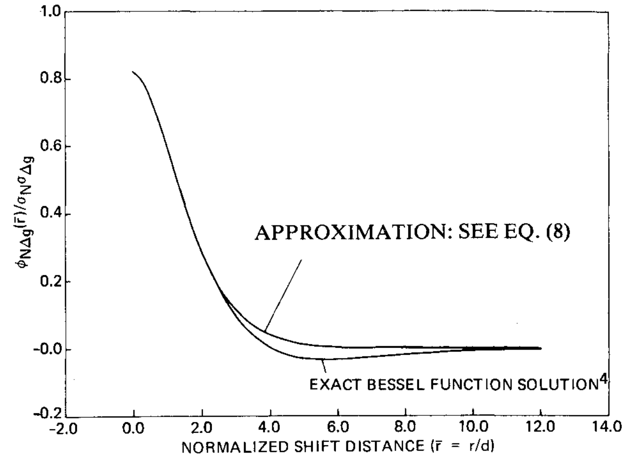

Fig. 2 Geoidal undulation-gravity anomaly cross-correlation approximation.

σ_ξ. This symmetry property provides an excellent check on the derivation of the correlation functions. All correlation functions are given in the Appendix.

Constant-Heading Time-Domain Form

In applying the third-order Markov undulation model to navigation problems, it is necessary to express the correlation functions in path-oriented variables (see Fig. 6) and convert to the time domain.

$$\lambda = \xi\cos\theta + \eta\sin\theta \qquad \mu = \eta\cos\theta - \xi\sin\theta \tag{11}$$

where λ, μ are the in- and cross-track components of vertical deflection.

Also, applying the similarity transform rules to the anomalous gravity gradients

$$\Gamma_{\lambda\lambda} = \Gamma_{yy}\cos^2\theta + 2\Gamma_{xy}\sin\theta\cos\theta + \Gamma_{xx}\sin^2\theta \tag{12}$$

$$\Gamma_{\lambda\mu} = \Gamma_{xy}(\cos^2\theta - \sin^2\theta) + (\Gamma_{xx} - \Gamma_{yy})\sin\theta\cos\theta \tag{13}$$

$$\Gamma_{\mu\mu} = \Gamma_{yy}\sin^2\theta - 2\Gamma_{xy}\sin\theta\cos\theta + \Gamma_{xx}\cos^2\theta \tag{14}$$

$$\Gamma_{\lambda z} = \Gamma_{yz}\cos\theta + \Gamma_{xz}\sin\theta \tag{15}$$

$$\Gamma_{\mu z} = \Gamma_{xz}\cos\theta - \Gamma_{yz}\sin\theta \tag{16}$$

Of course, N and Δg are unchanged by azimuth rotations. Then, it is possible to obtain the constant-heading form by evaluating all the correlation functions at $\theta = 0$, and making the associations

$$\begin{array}{ccc} N = N & \Delta g = \Delta g & \Gamma_{\mu\mu} = \Gamma_{xx} \\ \lambda = \xi & \Gamma_{\lambda\lambda} = \Gamma_{yy} & \Gamma_{\lambda z} = \Gamma_{yz} \\ \mu = \eta & \Gamma_{\lambda\mu} = \Gamma_{xy} & \Gamma_{\mu z} = \Gamma_{xz} \end{array}$$

The time-domain correlation functions are obtained from the space-domain functions by making the replacement $r = V|\tau|$, where V is the vehicle's speed and τ is the time-shift parameter. It is then possible to proceed directly to the shaping filter from the correlation matrix of auto- and cross-correlation functions expressed in the time domain.

Synthesis of Shaping Filter

Start with the correlation matrix

$$\phi(\tau) = \mathrm{E}[x(t)x^T(t+\tau)] \tag{17}$$

where, in our case, x is the state vector of anomalous gravity

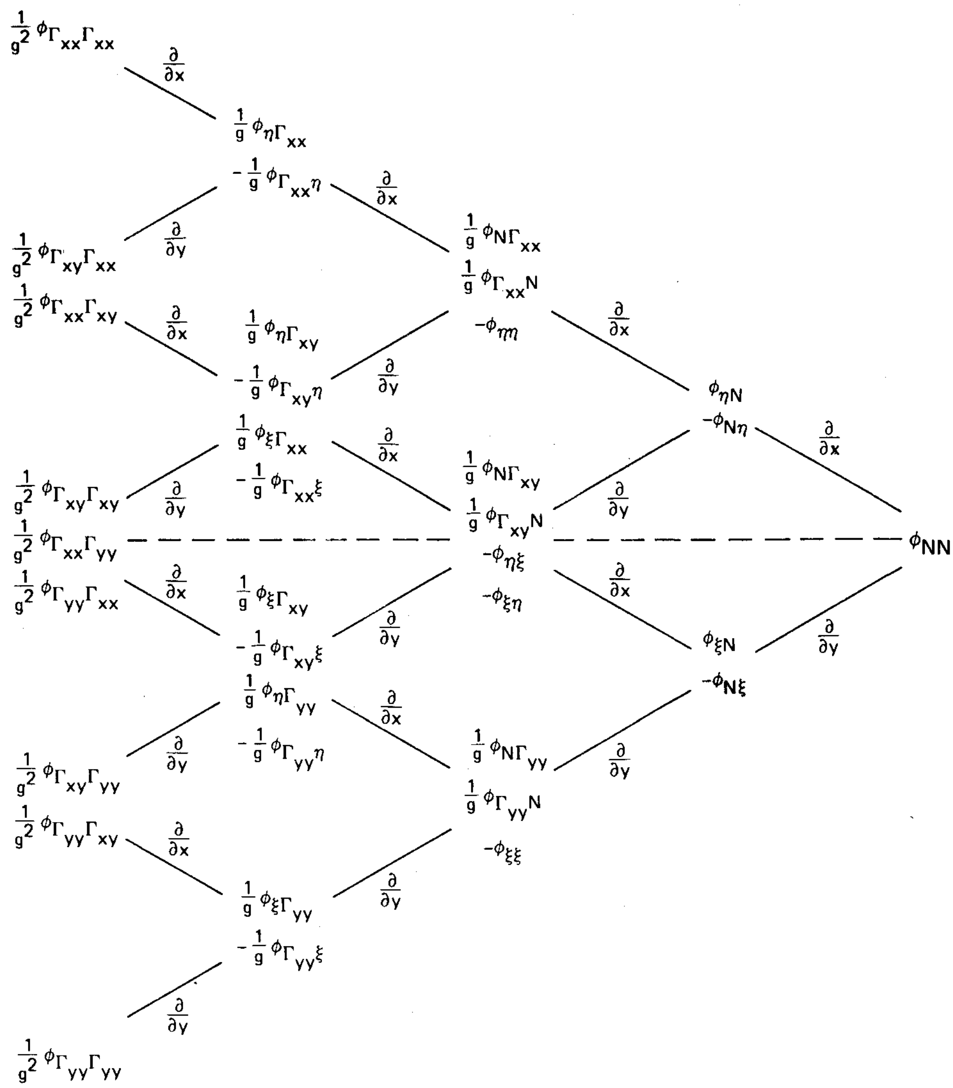

Fig. 3 Correlation functions derived from ϕ_{NN}.

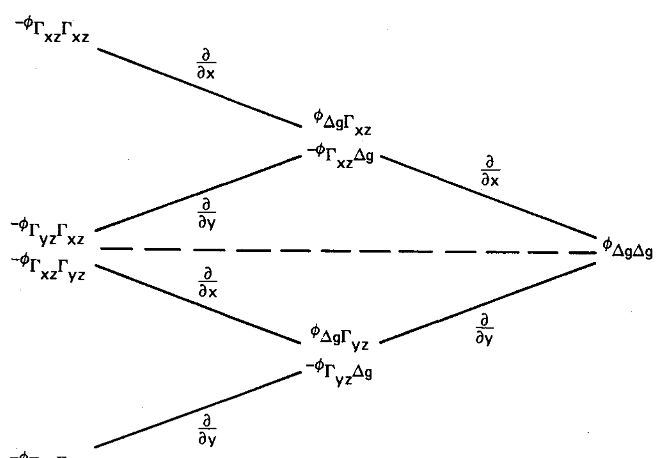

Fig. 4 Correlation functions derived from $\phi_{\Delta g\Delta g}$.

variables

$$x^T = (N \mid \lambda \mid \Gamma_{\lambda\lambda} \mid \Gamma_{\mu\mu} \mid \Delta g \mid \Gamma_{\lambda z} \mid \mu \mid \Gamma_{\lambda\mu} \mid \Gamma_{\mu z}) \tag{18}$$

It is desired to obtain an equation of the form

$$\dot{x}(t) = Fx(t) + w(t) \tag{19}$$

where F is the constant dynamics matrix having eigenvalues with negative real parts, and $w(t)$ is the vector of white noise inputs.

$$\mathrm{E}[w(t)w^T(t+\tau)] = Q\delta(\tau) \tag{20}$$

The relationship between x at a time t, and x at some future time $t+\tau$, is given by the solution to Eq. (19):

$$x(t+\tau) = \Phi(\tau)x(t) + \int_t^{t+\tau} \Phi(\tau-\sigma)w(\sigma)\mathrm{d}\sigma \tag{21}$$

where $\Phi(\tau)$ is the state-transition matrix defined by

$$\dot{\Phi}(\tau) = F\Phi(\tau) \qquad \Phi(0) = I \tag{22}$$

Using Eq. (21) in Eq. (17) gives

$$\phi(\tau) = \mathrm{E}[x(t)x^T(t)]\Phi^T(\tau) + \int_t^{t+\tau} \mathrm{E}[x(t)w^T(\sigma)]\Phi^T(\tau-\sigma)\mathrm{d}\sigma \tag{23}$$

The integral term in Eq. (23) is zero because the state at time t is uncorrelated with the input at time t, or any future time, $t \le \sigma \le \tau + t$. This leaves[6]

$$\phi(\tau) = P\Phi^T(\tau)$$

$$P = \mathrm{E}[x(t)x^T(t)] = \phi(0) = \text{covariance of } x \tag{24}$$

Then, at steady-state ($\dot{P} = 0$),

$$\dot{\phi}(\tau) = P\dot{\Phi}^T(\tau) = P\Phi^T(\tau)F^T \tag{25}$$

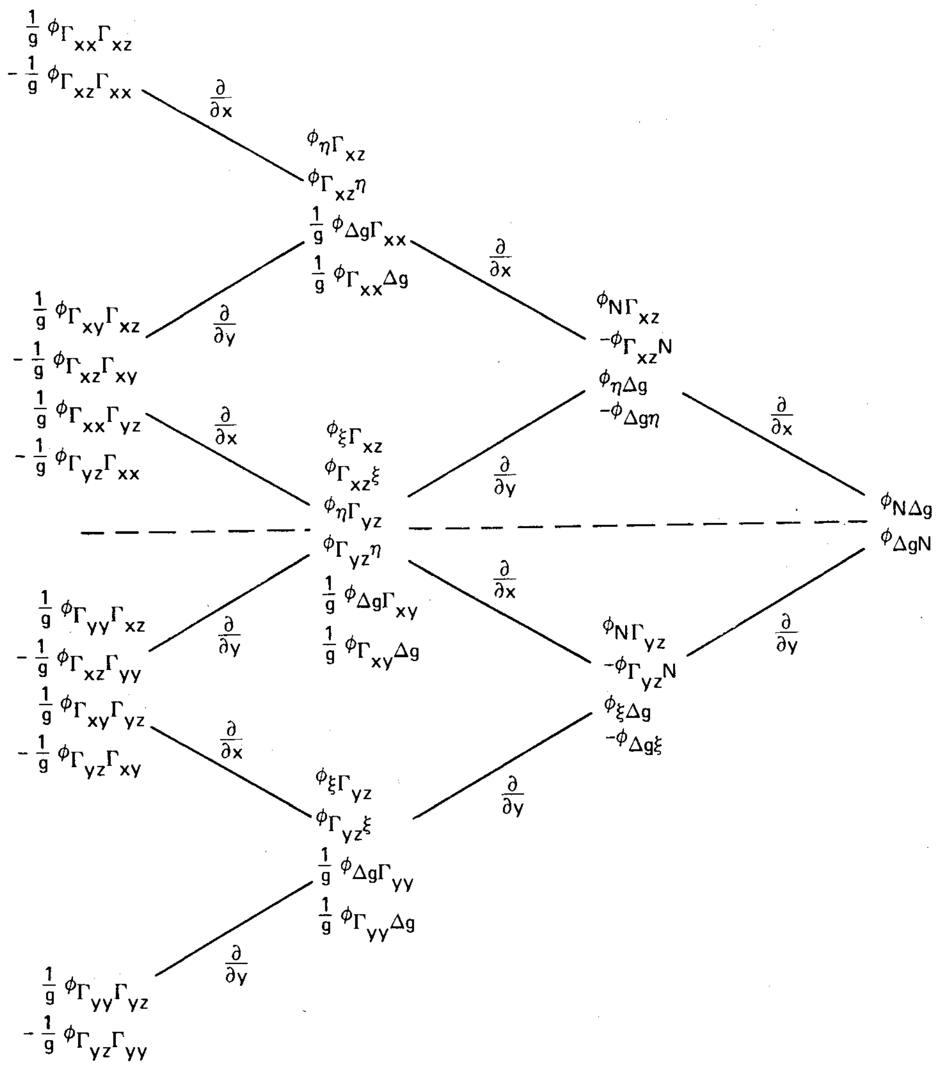

Fig. 5 Correlation functions derived from $\phi_{N\Delta g}$.

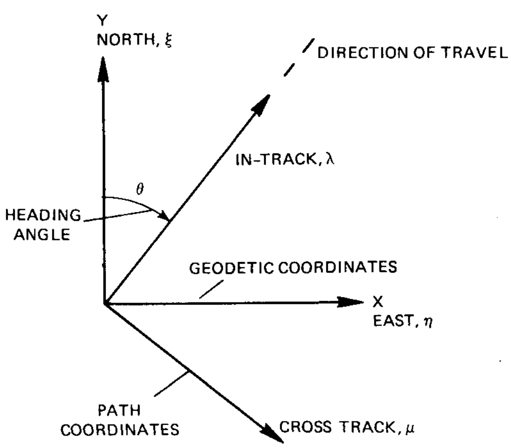

Fig. 6 Path-oriented variables.

$$
\begin{array}{cc}
 & \begin{array}{ccccccccc} N & \lambda & \Gamma_{\lambda\lambda} & \Gamma_{\mu\mu} & \Delta g & \Gamma_{\lambda z} & \mu & \Gamma_{\lambda\mu} & \Gamma_{\mu z} \end{array} \\
\underline{P} = \sigma_\eta^2 \begin{array}{c} N \\ \lambda \\ \Gamma_{\lambda\lambda} \\ \Gamma_{\mu\mu} \\ \Delta g \\ \Gamma_{\lambda z} \\ \mu \\ \Gamma_{\lambda\mu} \\ \Gamma_{\mu z} \end{array} &
\left[\begin{array}{ccccccccc}
3d^2 & 0 & -g & -g & 2dg & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & \frac{-3g}{2d} & 0 & 0 & 0 \\
-g & 0 & \frac{3g^2}{d^2} & \frac{g^2}{d^2} & \frac{-3g^2}{2d} & 0 & 0 & 0 & 0 \\
-g & 0 & \frac{g^2}{d^2} & \frac{3g^2}{d^2} & \frac{-3g^2}{2d} & 0 & 0 & 0 & 0 \\
2dg & 0 & \frac{-3g^2}{2d} & \frac{-3g^2}{2d} & 2g^2 & 0 & 0 & 0 & 0 \\
0 & \frac{-3g}{2d} & 0 & 0 & 0 & \frac{4g^2}{d^2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{-3g}{2d} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{g^2}{d^2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{-3g}{2d} & 0 & \frac{4g^2}{d^2}
\end{array}\right]
\end{array}
$$

Fig. 7 Covariance matrix.

Setting $\tau=0$, so that $\Phi(0)=\boldsymbol{I}$,

$$\dot{\boldsymbol{\phi}}(0)=\boldsymbol{P}\boldsymbol{F}^T \tag{26}$$

and solving gives the desired result of $\boldsymbol{F}$ in terms of $\boldsymbol{\phi}(\tau)$

$$\boldsymbol{F}=\dot{\boldsymbol{\phi}}^T(0)\boldsymbol{P}^{-1}=\dot{\boldsymbol{\phi}}^T(0)\boldsymbol{\phi}(0)^{-1} \tag{27}$$

$\boldsymbol{Q}$ may be determined from the steady-state form of the covariance propagation equation

$$\dot{\boldsymbol{P}}=\boldsymbol{0}=\boldsymbol{F}\boldsymbol{P}+\boldsymbol{P}\boldsymbol{F}^T+\boldsymbol{Q} \tag{28}$$

as

$$\boldsymbol{Q}=-\boldsymbol{F}\boldsymbol{P}-\boldsymbol{P}\boldsymbol{F}^T \tag{29}$$

or

$$\boldsymbol{Q}=-\dot{\boldsymbol{\phi}}^T(0)-\dot{\boldsymbol{\phi}}(0) \tag{30}$$

Thus, it is possible to find the $\boldsymbol{F}$ matrix from Eq. (27). The $\boldsymbol{Q}$ matrix can be found from Eq. (30).

	N	λ	$\Gamma_{\lambda\lambda}$	$\Gamma_{\mu\mu}$	Δg	$\Gamma_{\lambda z}$	μ	$\Gamma_{\lambda\mu}$	$\Gamma_{\mu z}$
N	0	$-V$	0	0	0	0	0	0	0
λ	0	0	$\frac{-V}{g}$	0	0	0	0	0	0
$\Gamma_{\lambda\lambda}$	$\frac{4gV}{d^3}$	$\frac{-51gV}{28d^2}$	$\frac{-11V}{2d}$	$\frac{-5V}{2d}$	$\frac{-10V}{d^2}$	$\frac{-45V}{14d}$	0	0	0
$\Gamma_{\mu\mu}$	$\frac{2gV}{d^3}$	$\frac{-17gV}{28d^2}$	$\frac{-7V}{4d}$	$\frac{-9V}{4d}$	$\frac{-5V}{d^2}$	$\frac{-15V}{14d}$	0	0	0
Δg	0	0	0	0	0	V	0	0	0
$\Gamma_{\lambda z}$	$\frac{gV}{2d^3}$	$\frac{-60gV}{7d^2}$	$\frac{59V}{16d}$	$\frac{5V}{16d}$	$\frac{V}{2d^2}$	$\frac{-40V}{7d}$	0	0	0
μ	0	0	0	0	0	0	0	$\frac{-V}{g}$	0
$\Gamma_{\lambda\mu}$	0	0	0	0	0	0	$\frac{-17gV}{28d^2}$	$\frac{-2V}{d}$	$\frac{-15V}{14d}$
$\Gamma_{\mu z}$	0	0	0	0	0	0	$\frac{-30gV}{7d^2}$	$\frac{27V}{8d}$	$\frac{-20V}{7d}$

Fig. 8 Dynamics matrix.

$$\underline{Q} = \sigma_\eta^2 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{16g^2V}{d^3} & \frac{4g^2V}{d^3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4g^2V}{d^3} & \frac{6g^2V}{d^3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{20g^2V}{d^3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4g^2V}{d^3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{10g^2V}{d^3} \end{bmatrix}$$

Fig. 9 *Q* matrix.

Resulting Linear System

The ***P***, ***F***, and ***Q*** matrices are given in Figs. 7-9. In the dynamics matrix ***F***, all the geodetic relations in Eqs. (3-6) are satisfied. Note that the cross-track variables μ, $\Gamma_{\lambda\mu}$, and $\Gamma_{\mu z}$ are not coupled to the in-track variables.

If the anomalous gravity field variables are to be expressed in geodetic coordinates, the inverse transforms of Eqs. (11-16) can be applied to the outputs.

Conclusions

The model provides a fully self-consistent shaping filter for covariance analysis, Kalman filtering, optimal prediction, and smoothing problems involving gravity-sensitive instrumentation moving along straight, horizontal tracks. It can be applied to covariance analysis and optimal estimation of inertial navigation or gravity gradiometry system performance. The technique used for shaping filter synthesis provides a systematic route to the linear system form of the Markov model and should be useful for developing more sophisticated gravity models or multidimensional Markov models for other problems.

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Appendix: Auto- and Cross-Correlation Functions of the Third-Order Markov Undulation Model

Relations Between Rms Values

$$\sigma_N = \sqrt{3}d\sigma_\eta \qquad \sigma_\xi = \sigma_\eta \qquad \sigma_{\Delta g} = \sqrt{2}g\sigma_\eta$$

$$\sigma_{\Gamma_{xx}} = \sigma_{\Gamma_{yy}} = (\sqrt{3}g/d)\,\sigma_\eta \qquad \sigma_{\Gamma_{xy}} = (g/d)\,\sigma_\eta \qquad \sigma_{\Gamma_{xz}} = \sigma_{\Gamma_{yz}} = (2g/d)\,\sigma_\eta$$

Auto- and Cross-Correlation Functions

$$\phi_{NN}(r) = \sigma_N^2\left(1 + \frac{r}{d} + \frac{r^2}{3d^2}\right)e^{-r/d}$$

$$\phi_{N\eta}(r,\theta) = -\phi_{\eta N}(r,\theta) = \sigma_N\sigma_\eta\frac{\sqrt{3}}{3}\left(\frac{r}{d} + \frac{r^2}{d^2}\right)e^{-r/d}\sin\theta$$

$$\phi_{N\xi}(r,\theta) = -\phi_{\xi N}(r,\theta) = \sigma_N\sigma_\xi\frac{\sqrt{3}}{3}\left(\frac{r}{d} + \frac{r^2}{d^2}\right)e^{-r/d}\cos\theta$$

$$\phi_{N\Delta g}(r) = \phi_{\Delta g N}(r) \cong \sigma_N\sigma_{\Delta g}\frac{\sqrt{6}}{3}\left(1 + \frac{3r}{2d} + \frac{3r^2}{4d^2}\right)e^{-3r/2d}$$

$$\phi_{N\Gamma_{xx}}(r,\theta)=\phi_{\Gamma_{xx}N}(r,\theta)=-\frac{\sigma_N\sigma_{\Gamma_{xx}}}{3}\left(1+\frac{r}{d}-\frac{r^2}{d^2}\sin^2\theta\right)e^{-r/d}$$

$$\phi_{N\Gamma_{xy}}(r,\theta)=\phi_{\Gamma_{xy}N}(r,\theta)=\sigma_N\sigma_{\Gamma_{xy}}\frac{\sqrt{3}r^2}{3d^2}e^{-r/d}\sin\theta\cos\theta$$

$$\phi_{N\Gamma_{yy}}(r,\theta)=\phi_{\Gamma_{yy}N}(r,\theta)=-\frac{\sigma_N\sigma_{\Gamma_{yy}}}{3}\left(1+\frac{r}{d}-\frac{r^2}{d^2}\cos^2\theta\right)e^{-r/d}$$

$$\phi_{N\Gamma_{xz}}(r,\theta)=-\phi_{\Gamma_{xz}N}(r,\theta)\cong-\sigma_N\sigma_{\Gamma_{xz}}\frac{\sqrt{3}}{4}\left(\frac{r}{d}+\frac{3r^2}{2d^2}\right)e^{-3r/2d}\sin\theta$$

$$\phi_{N\Gamma_{yz}}(r,\theta)=-\phi_{\Gamma_{yz}N}(r,\theta)\cong-\sigma_N\sigma_{\Gamma_{yz}}\frac{\sqrt{3}}{4}\left(\frac{r}{d}+\frac{3r^2}{2d^2}\right)e^{-3r/2d}\cos\theta$$

$$\phi_{\eta\eta}(r,\theta)=\sigma_\eta^2\left(1+\frac{r}{d}-\frac{r^2}{d^2}\sin^2\theta\right)e^{-r/d}$$

$$\phi_{\eta\xi}(r,\theta)=\phi_{\xi\eta}(r,\theta)=-\sigma_\eta\sigma_\xi\frac{r^2}{d^2}e^{-r/d}\sin\theta\cos\theta$$

$$\phi_{\eta\Delta g}(r,\theta)=-\phi_{\Delta g\eta}(r,\theta)\cong-\sigma_\eta\sigma_{\Delta g}\frac{3\sqrt{2}}{4}\left(\frac{r}{d}+\frac{3r^2}{2d^2}\right)e^{-3r/2d}\sin\theta$$

$$\phi_{\eta\Gamma_{xx}}(r,\theta)=-\phi_{\Gamma_{xx}\eta}(r,\theta)=\sigma_\eta\sigma_{\Gamma_{xx}}\frac{\sqrt{3}}{3}\left(\frac{3r}{d}\sin\theta-\frac{r^2}{d^2}\sin^3\theta\right)e^{-r/d}$$

$$\phi_{\eta\Gamma_{xy}}(r,\theta)=-\phi_{\Gamma_{xy}\eta}(r,\theta)=\sigma_\eta\sigma_{\Gamma_{xy}}\left(\frac{r}{d}\cos\theta-\frac{r^2}{d^2}\cos\theta\sin^2\theta\right)e^{-r/d}$$

$$\phi_{\eta\Gamma_{yy}}(r,\theta)=-\phi_{\Gamma_{yy}\eta}(r,\theta)=\sigma_\eta\sigma_{\Gamma_{yy}}\frac{\sqrt{3}}{3}\left(\frac{r}{d}\sin\theta-\frac{r^2}{d^2}\sin\theta\cos^2\theta\right)e^{-r/d}$$

$$\phi_{\eta\Gamma_{xz}}(r,\theta)=\phi_{\Gamma_{xz}\eta}(r,\theta)\cong-\sigma_\eta\sigma_{\Gamma_{xz}}\frac{3}{4}\left(1+\frac{3r}{2d}-\frac{9r^2}{4d^2}\sin^2\theta\right)e^{-3r/2d}$$

$$\phi_{\eta\Gamma_{yz}}(r,\theta)=\phi_{\Gamma_{yz}\eta}(r,\theta)\cong\sigma_\eta\sigma_{\Gamma_{yz}}\frac{27r^2}{16d^2}e^{-3r/2d}\sin\theta\cos\theta$$

$$\phi_{\xi\xi}(r,\theta)=\sigma_\xi^2\left(1+\frac{r}{d}-\frac{r^2}{d^2}\cos^2\theta\right)e^{-r/d}$$

$$\phi_{\xi\Delta g}(r,\theta)=-\phi_{\Delta g\xi}(r,\theta)\cong-\frac{3\sqrt{2}\sigma_\xi\sigma_{\Delta g}}{4}\left(\frac{r}{d}+\frac{3r^2}{2d^2}\right)e^{-3r/2d}\cos\theta$$

$$\phi_{\xi\Gamma_{xx}}(r,\theta)=-\phi_{\Gamma_{xx}\xi}(r,\theta)=\sigma_\xi\sigma_{\Gamma_{xx}}\frac{\sqrt{3}}{3}\left(\frac{r}{d}\cos\theta-\frac{r^2}{d^2}\cos\theta\sin^2\theta\right)e^{-r/d}$$

$$\phi_{\xi\Gamma_{xy}}(r,\theta)=-\phi_{\Gamma_{xy}\xi}(r,\theta)=\sigma_\xi\sigma_{\Gamma_{xy}}\left(\frac{r}{d}\sin\theta-\frac{r^2}{d^2}\sin\theta\cos^2\theta\right)e^{-r/d}$$

$$\phi_{\xi\Gamma_{yy}}(r,\theta)=-\phi_{\Gamma_{yy}\xi}(r,\theta)=\sigma_\xi\sigma_{\Gamma_{yy}}\frac{\sqrt{3}}{3}\left(\frac{3r}{d}\cos\theta-\frac{r^2}{d^2}\cos^3\theta\right)e^{-r/d}$$

$$\phi_{\xi\Gamma_{xz}}(r,\theta)=\phi_{\Gamma_{xz}\xi}(r,\theta)\cong\sigma_\xi\sigma_{\Gamma_{xz}}\frac{27r^2}{16d^2}e^{-3r/2d}\sin\theta\cos\theta$$

$$\phi_{\xi\Gamma_{yz}}(r,\theta)=\phi_{\Gamma_{yz}\xi}(r,\theta)\cong-\sigma_\xi\sigma_{\Gamma_{yz}}\frac{3}{4}\left(1+\frac{3r}{2d}-\frac{9r^2}{4d^2}\cos^2\theta\right)e^{-3r/2d}$$

$$\phi_{\Delta g\Delta g}(r,\theta)=\sigma_{\Delta g}^2\left(1+\frac{r}{d}-\frac{r^2}{2d^2}\right)e^{-r/d}$$

$$\phi_{\Delta g\Gamma_{xx}}(r,\theta)=\phi_{\Gamma_{xx}\Delta g}(r,\theta)\cong-\sigma_{\Delta g}\sigma_{\Gamma_{xx}}\frac{\sqrt{6}}{4}\left(1+\frac{3r}{2d}-\frac{9r^2}{4d^2}\sin^2\theta\right)e^{-3r/2d}$$

$$\phi_{\Delta g\Gamma_{xy}}(r,\theta)=\phi_{\Gamma_{xy}\Delta g}(r,\theta)\cong\sigma_{\Delta g}\sigma_{\Gamma_{xy}}\frac{27\sqrt{2}r^2}{16d^2}e^{-3r/2d}\sin\theta\cos\theta$$

$$\phi_{\Delta g\Gamma_{yy}}(r,\theta)=\phi_{\Gamma_{yy}\Delta g}(r,\theta)\cong-\sigma_{\Delta g}\sigma_{\Gamma_{yy}}\frac{\sqrt{6}}{4}\left(1+\frac{3r}{2d}-\frac{9r^2}{4d^2}\cos^2\theta\right)e^{-3r/2d}$$

$$\phi_{\Delta g\Gamma_{xz}}(r,\theta)=-\phi_{\Gamma_{xz}\Delta g}(r,\theta)=-\sigma_{\Delta g}\sigma_{\Gamma_{xz}}\sqrt{2}\left(\frac{r}{d}-\frac{r^2}{4d^2}\right)e^{-r/d}\sin\theta$$

$$\phi_{\Delta g\Gamma_{yz}}(r,\theta)=-\phi_{\Gamma_{yz}\Delta g}(r,\theta)=-\sigma_{\Delta g}\sigma_{\Gamma_{yz}}\sqrt{2}\left(\frac{r}{d}-\frac{r^2}{4d^2}\right)e^{-r/d}\cos\theta$$

$$\phi_{\Gamma_{xx}\Gamma_{xx}}(r,\theta)=\sigma^2_{\Gamma_{xx}}\left(1-\frac{r}{3d}(6\sin^2\theta-\sin^4\theta)+\frac{r^2}{3d^2}\sin^4\theta\right)e^{-r/d}$$

$$\phi_{\Gamma_{xx}\Gamma_{xy}}(r,\theta)=\phi_{\Gamma_{xy}\Gamma_{xx}}(r,\theta)=\sigma_{\Gamma_{xx}}\sigma_{\Gamma_{xy}}\frac{\sqrt{3}}{3}\left(\frac{r}{d}(-3\sin\theta\cos\theta+\cos\theta\sin^3\theta)+\frac{r^2}{d^2}\cos\theta\sin^3\theta\right)e^{-r/d}$$

$$\phi_{\Gamma_{xx}\Gamma_{yy}}(r,\theta)=\phi_{\Gamma_{yy}\Gamma_{xx}}(r,\theta)=\frac{\sigma_{\Gamma_{xx}}\sigma_{\Gamma_{yy}}}{3}\left(1-\frac{r}{d}(1-\sin^2\theta\cos^2\theta)+\frac{r^2}{d^2}\sin^2\theta\cos^2\theta\right)e^{-r/d}$$

$$\phi_{\Gamma_{xx}\Gamma_{xz}}(r,\theta)=-\phi_{\Gamma_{xz}\Gamma_{xx}}(r,\theta)\cong\sigma_{\Gamma_{xx}}\sigma_{\Gamma_{xz}}\frac{27\sqrt{3}}{16}\left(\frac{r}{d}\sin\theta-\frac{r^2}{2d^2}\sin^3\theta\right)e^{-3r/2d}$$

$$\phi_{\Gamma_{xx}\Gamma_{yz}}(r,\theta)=-\phi_{\Gamma_{xx}\Gamma_{xz}}(r,\theta)\cong\sigma_{\Gamma_{xx}}\sigma_{\Gamma_{yz}}\frac{9\sqrt{3}}{16}\left(\frac{r}{d}\cos\theta-\frac{3r^2}{2d^2}\cos\theta\sin^2\theta\right)e^{-3r/2d}$$

$$\phi_{\Gamma_{xy}\Gamma_{xy}}(r,\theta)=\sigma^2_{\Gamma_{xy}}\left(1-\frac{r}{d}\left(1-\sin^2\theta\cos^2\theta\right)+\frac{r^2}{d^2}\sin^2\theta\cos^2\theta\right)e^{-r/d}$$

$$\phi_{\Gamma_{xy}\Gamma_{yy}}(r,\theta)=\phi_{\Gamma_{yy}\Gamma_{xy}}(r,\theta)=\sigma_{\Gamma_{xy}}\sigma_{\Gamma_{yy}}\frac{\sqrt{3}}{3}\left(\frac{r}{d}(-3\sin\theta\cos\theta+\sin\theta\cos^3\theta)+\frac{r^2}{d^2}\sin\theta\cos^3\theta\right)e^{-r/d}$$

$$\phi_{\Gamma_{xy}\Gamma_{xz}}(r,\theta)=-\phi_{\Gamma_{xz}\Gamma_{xy}}(r,\theta)\cong\frac{27\sigma_{\Gamma_{xy}}\sigma_{\Gamma_{yz}}}{16}\left(\frac{r}{d}\cos\theta-\frac{3r^2}{2d^2}\cos\theta\sin^2\theta\right)e^{-3r/2d}$$

$$\phi_{\Gamma_{xy}\Gamma_{yz}}(r,\theta)=-\phi_{\Gamma_{yz}\Gamma_{xy}}(r,\theta)\cong\frac{27\sigma_{\Gamma_{xy}}\sigma_{\Gamma_{yz}}}{16}\left(\frac{r}{d}\sin\theta-\frac{3r^2}{2d^2}\sin\theta\cos^2\theta\right)e^{-3r/2d}$$

$$\phi_{\Gamma_{yy}\Gamma_{yy}}(r,\theta)=\sigma^2_{\Gamma_{yy}}\left(1-\frac{r}{3d}(6\cos^2\theta-\cos^4\theta)+\frac{r^2}{3d^2}\cos^4\theta\right)e^{-r/d}$$

$$\phi_{\Gamma_{yy}\Gamma_{xz}}(r,\theta)=-\phi_{\Gamma_{xz}\Gamma_{yy}}(r,\theta)\cong\sigma_{\Gamma_{yy}}\sigma_{\Gamma_{xz}}\frac{9\sqrt{3}}{16}\left(\frac{r}{d}\sin\theta-\frac{3r^2}{2d^2}\sin\theta\cos^2\theta\right)e^{-3r/2d}$$

$$\phi_{\Gamma_{yy}\Gamma_{yz}}(r,\theta)=-\phi_{\Gamma_{zy}\Gamma_{yy}}(r,\theta)\cong\sigma_{\Gamma_{yy}}\sigma_{\Gamma_{yz}}\frac{27\sqrt{3}}{16}\left(\frac{r}{d}\cos\theta-\frac{r^2}{2d^2}\cos^3\theta\right)e^{-3r/2d}$$

$$\phi_{\Gamma_{xz}\Gamma_{xz}}(r,\theta)=\sigma^2_{\Gamma_{xz}}\left(1-\frac{r}{4d}(5\sin^2\theta+1)+\frac{r^2}{4d^2}\sin^2\theta\right)e^{-r/d}$$

$$\phi_{\Gamma_{xz}\Gamma_{yz}}(r,\theta)=\phi_{\Gamma_{yz}\Gamma_{xz}}(r,\theta)=-\frac{\sigma_{\Gamma_{xz}}\sigma_{\Gamma_{yz}}}{4}\left(\frac{5r}{d}-\frac{r^2}{d^2}\right)e^{-r/d}\sin\theta\cos\theta$$

$$\phi_{\Gamma_{yz}\Gamma_{yz}}(r,\theta)=\sigma^2_{\Gamma_{yz}}\left(1-\frac{r}{4d}(5\cos^2\theta+1)+\frac{r^2}{4d^2}\cos^2\theta\right)e^{-r/d}$$